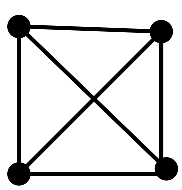
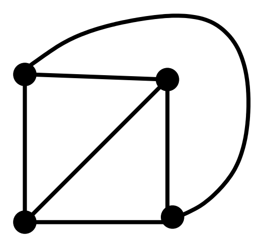
Kuratowski Notes
18.310, Fall 2005, Prof. Peter Shor
Revised Fall 2007

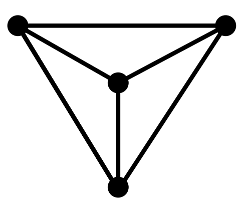
Unfortunately, the OCW notes on Kuratowski's theorem seem to have several things substantially wrong with the proof, and the notes from Prof. Kleitman's website are too vague to be able to deduce the proof from them. I'm just going to type in the OCW notes, changing things to make the proofs correct.

We can, given a graph, attempt to draw it on a piece of paper, representing its vertices by points and its edges by either straight lines or nice curves. We then define a graph G to be planar if it can be so drawn without any of the curves or lines that represent its edges crossing one another or meeting a third vertex on the way from one vertex to the other. Note that there might be many ways to draw the graph such that its edges cross, but as long as there is some way to draw it such that no edges cross, then it is planar. For example, looking at the graph in the figure below, you may not think that it is planar because two of its edges cross. However, the same graph can be drawn in a different way such that no edges cross and we see that it is in fact planar.



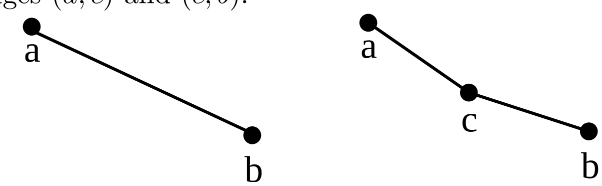


One obvious question is whether we need to use curves for planar graphs, or whether all planar graphs can be drawn with straight line segments. It turns out that any planar graph can be drawn in the plane with straight line segments for its edges. For example, the graph above can be drawn with straight lines as follows.

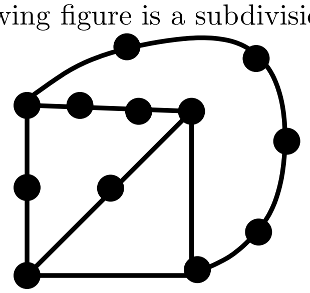


We will define the *degree* of a vertex to be the number of edges that contain it as an endpoint.

It will be useful later to have a notion of *adding a vertex in the middle of an edge*. Adding a vertex in the middle of an edge means replacing an edge (a, b) by two new edges (a, c) and (c, b).



We will say that a *subdivision* of G is any graph that is obtainable by adding repeatedly adding vertices in the middle of edges of G. (For the sake of completeness, we will consider G to be a subdivision of itself). For example, the following figure is a subdivision of the graph above.



Note that this definition is different from the way Prof. Kleitman defined subdivision in the OCW notes. What he calls a subdivision is generally called a *minor*, and what I call a subdivision is generally called a subdivision.[1] Kuratowski's theorem is true for both minors and for subdivisions.

Another obvious question is: is there a convenient way to characterize planar graphs? Indeed there is. Before discussing and proving it we make some remarks, which we will prove.

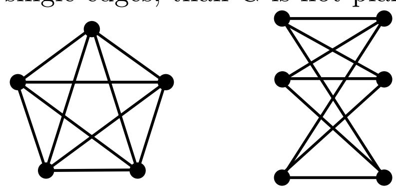
1. There are two fairly small graphs which are not planar, K_5 and $K_{3,3}$.

2. We can add vertices in the middle of any of these two graphs as we like, and that will not help to make them planar. Adding a vertex in the middle of an edge here means replacing an edge (a, b) by two new edges (a, c) and (c, b).

3. Every graph which contains as a subgraph either K_5 or $K_{3,3}$ or a graph obtained from these by adding vertices in the middle of edges cannot be planar.

[1]Actually, when Prof. Kleitman says graph G is a subdivision of H, the accepted terminology is that H is a minor of G. A minor of G is a graph that can be obtained from G by deleting and contracting edges; here contracting means deleting the edge and merging its two endpoints into a single vertex.

This means that if we can obtain one of the graphs K_5 or a $K_{3,3}$ from G by deleting (throwing away) some edges, and replacing long paths in the resulting graph by single edges, than G is not planar.



K_5 $K_{3,3}$

We will now prove these statements. The first follows from this remark. If we take any drawing of either K_5 (or more generally K_{2j+1}) or $K_{3,3}$ (or more generally $K_{2k+1,2j+1}$) in the plane) the number of crossings between edges *whose vertex sets are disjoint*[2] has the same value mod 2 in each drawing (we count a point of tangency between two edges as either 2 or 0 crossings).

It follows immediately from this statement that if we can find a drawing of either K_5 or $K_{3,3}$ with an odd number of crossings between edges whose vertex sets are disjoint, it cannot be drawn with any even number of crossings, including zero.

So let us prove the statement.

We start with two drawings of the same graph, with vertex sets the same for each. We will take each edge of the first one at a time and slowly and continuously move it until it reaches the position of the same edge in he second drawings. When we are done, the two drawings will be identical, so they will have the same number of crossings.

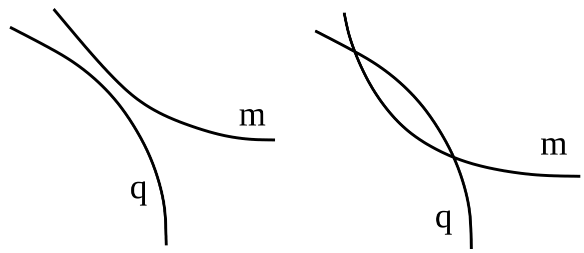
To prove the result we notice that the number of crossings of the moved edge with any other edge having different endpoints can only change by an even number.

How could it change?

If the edge m being moved does not either become tangent to another edge q or cross over one of the endpoints of another edge, the number of crossings between m and q will not change in any way. The crossings, if any, will merely slide along q.

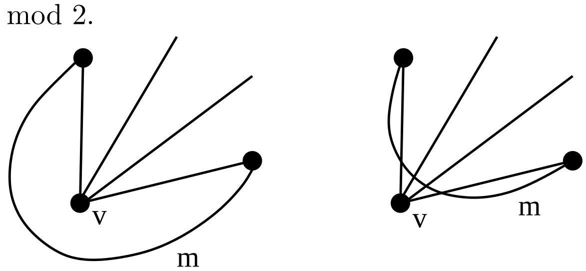
When m and q become tangent and then cross, or become tangent and uncross, the number of crossings between m and q will change by 2.

[2]The reason we have to count crossings with disjoint endpoints is because if two edges share an endpoint, then we could make them cross each other an arbitrary number of times if we wanted to. If we are using straight-line drawings, these edges never cross at all.



When m crosses over an endpoint v, the number of crossings of m with every edge containing v changes by 1, either up or down.

In the case of K_{2j+1}, since every vertex shares an edge with every other vertex, two of these crossings will involve edges that share endpoints with the two ends of m. The number of crossings not counting these two (since these edges do not have disjoint vertex sets) will therefore change by an even number when m passes over v (since the vertices of K_{2j+1} have even degree), which is 0 mod 2.



We conclude that the number of crossings in either case can never change mod 2 as the first drawing is transformed into the second one. Thus, the number of crossings mod 2 must have been the same to begin with, which is what we sent out to prove Q.E.D.

We turn then to the question posed above of how to characterize planar graphs. It turns out that there is a very nice theorem, called Kuratowski's Theorem, which tells us exactly when a graph is non-planar

A graph is planar if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.

This theorem tells us that the absence of these two configurations and their subdivisions, which we have seen are enough to ruin planarity, is enough to ensure planarity.

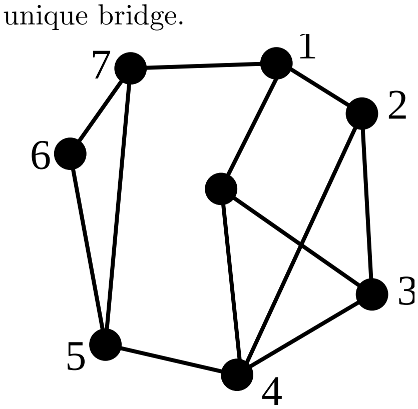
We will now give a the proof of a special case of Kuratowski's theorem, as well as the outline of a complete proof. The special case is the case where G has maximum degree 3. In this case, Kuratowski's theorem says that G must contain a subdivision of a $K_{3,3}$, since K_5 has vertices of degree 4, and we can never obtain a degree 4 vertex by subdivision of a graph of maximum degree 3.

In the presentation of the proof, I will try to do two things at the same time. We will start out presenting arguments that apply not only to maximum degree 3 graphs but also higher degree graphs. In fact Kuratowski's theorem in general can be proved by continuing this line of argument. However, the details seem to get pretty complicated, and at the point when they start getting difficult, we'll switch to talking about just the case of maximum degree 3 graphs so you can understand how the proof goes without wading through all the messy details. I'll give some exercises at the end of these notes which will show how to extend the proof to the general case.

Recall that we defined a cycle in a graph as a path whose endpoint is the same as its starting point, but which does not repeat any vertex in between. This is often called a simple cycle.

Recall that a *chord* of a cycle C in a graph G was an edge (not in C) between two vertices of C.

Before we can go on, we need to define a *bridge* of a cycle C in a graph G. A bridge is a maximal[3] set of edges in G connected by vertices not in C. In other words, it is a set of edges connected to each other by vertices not on C, and connected to the rest of the graph G only by vertices of C. For example, the graph pictured below has three bridges, two of them chords (edges 2-4 and 5-7) and one of them having three edges (connecting 1,3, and 4). Every non-cycle edge is in a unique bridge.



G

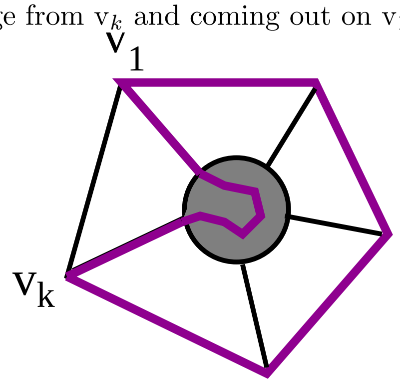
Our first observation is that it is enough to prove Kuratowski's theorem for graphs G with no vertices of degree 2. If a graph G has vertices of degree 2, we can remove all of these and merge their adjoining edges to obtain a

[3]Maximal means that it can't be made any larger, not that it has the most edges of any such set. This means any edge adjacent to a non-cycle vertex in a bridge must be in the bridge.

new graph G′ with no vertices of degree 2. G is non-planar if and only if G′ is, and G has a $K_{3,3}$ or K_5 subdivision if and only if G′ does, so if Kuratowski's theorem holds for G′ it also holds for G.

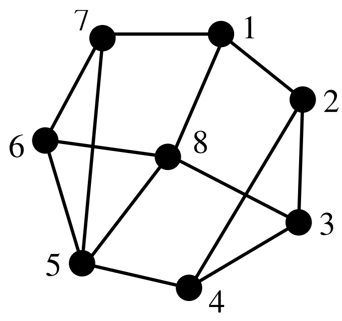
Suppose that G is a minimal non-planar graph. This is a graph such that if you remove one edge then it is planar (we can start with any non-planar graph and remove edges until it is minimal in this sense).

Our next step is to shat that G contains a cycle C with at least two bridges. One way to do this is to consider the longest cycle C in G. Suppose that it has only one bridge. If this bridge were a chord, then G would be planar, so it's not a chord. Because G has no degree 2 vertices, this bridge must connect all to the vertices in C. Since it's not a chord, we can find a longer cycle by taking a path that goes through the bridge (see figure). By the definition of a bridge, all the edges in the bridge are connected via non-cycle vertices, so we can find a longer cycle C′ by first going around the cycle C, then going into the bridge from v_k and coming out on v_1.



We will call the vertices connecting a bridge with our cycle C the *feet* of this bridge.

We say that two bridges are "compatible" if they can both be drawn inside C without edge crossings, and "incompatible" otherwise. For example, in the figure below, the chordal bridges 2-4 and 5-7 are compatible, and the non-chordal bridge 1-3-5-6-8 is incompatible with them both. Two incompatible bridges cannot both be drawn inside (or outside) the cycle C without crossing.



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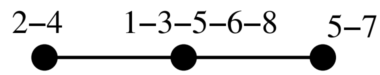
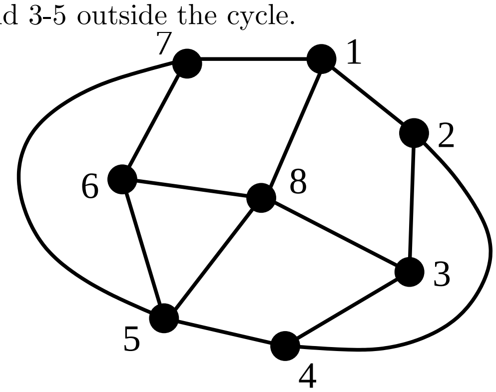


bridge graph

We now need to prove that if we have a set of bridges of C, all pairs of which are compatible, then all of these bridges can be drawn inside (or all outside) the cycle C without any crossings. This can be proved formally by induction, by showing that there is one bridge which is "inside" all the other bridges; i.e., all the other bridges are on the same side of it.

We next define a *bridge graph.* Its vertices are the bridges, and there is an edge connnecting any pair of bridges that are incompatible. For an example, both the graph G and its bridge graph are shown in the figure above.

If we take a pair of incompatible bridges, we can draw one of them inside the cycle and the other outside the cycle without any crossings. If we could do this for every pair of incompatible bridges, then we could draw G with no crossings and it would be planar. This means that if there was a way of dividing the bridge graph into two groups of vertices so that there is no edge between any of the vertices of a group, then G is planar. The graph in the previous figure is shown drawn as a planar graph here, by moving the two bridges 1-3 and 3-5 outside the cycle.



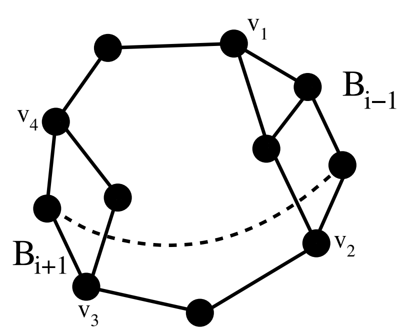
G

We thus have that if G is non-planar, the bridge graph is not bipartite.

We saw earlier that if a graph is not bipartite, then it contains an odd cycle. In fact, it contains a chordless odd cycle. This means that the bridge graph of G must contain a chordless odd cycle.[4] Thus, we can prove Kuratowski's theorem if we can show that any chordless odd cycle in the bridge graph requires a configuration in G that is obtained by subdividing edges from K_5 or $K_{3,3}$.

At this point, to make things easier, let's restrict all vertices of G to have degree 3. For this case, we are able to show that we can assume that all bridges of our cycle C must be chords.

Let's show this now. We'll concentrate on one bridge that is not a chord, that we'll call B_i, and show that if it is not a chord, then we can replace it by a chord between two of its feet and still have a non-planar graph. There are two cases. First, we'll assume that the chordless odd cycle in our bridge graph is not a triangle. Then, because the bridge graph was an odd cycle, there are exactly two bridges that are incompatible with B_i. Call them B_{i-1} and B_{i+1}. Because they are compatible, and because G has degree three, they must look something like the following figure. Specifically, the feet of B_{i-1} are between two vertices v_1 and v_2, and the feet of B_{i+1} are between two vertices v_3 and v_4, where v_1, v_2, v_3 and v_4 appear in that order on the cycle C[5].



G

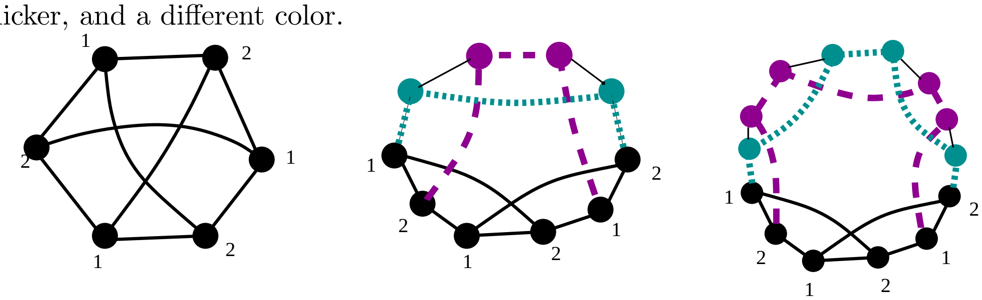
Now, for B_i to be incompatible with both bridges B_{i-1} and B_{i+1}, it must have at least one foot between v_1 and v_2 in the above figure, and at least one other between v_3 and v_4. But then, if we replace B_i by the chord that joins these two feet, we get a subgraph of G which is also non-planar because the bridge graph is also an odd cycle, so G couldn't have been minimal, a contradiction.

[4]At this point, there are two cycles around, the cycle C of G, and the chordless odd cycle in our bridge graph. Try not to confuse them.

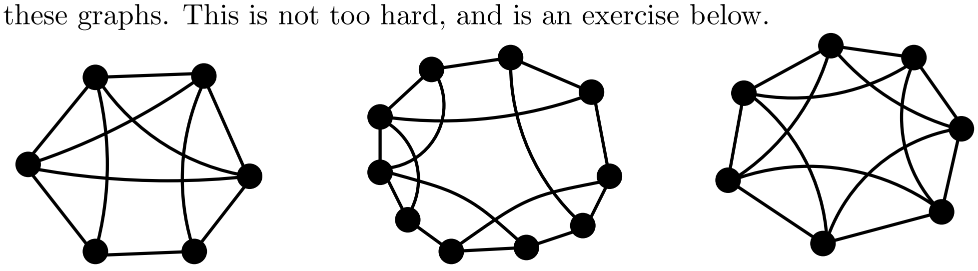
[5]In the degree > 3 case, we may have $v_2 = v_3$ and $v_4 = v_1$.

The other case is when the bridge graph is a triangle. In this case, the two bridges B_{i-1} and B_{i+1} must now be incompatible. Again, although it's a little tricker, it is possible to show that we can replace B_i by a chord joining two of its feet to find a subgraph of G which is also non-planar. (Exercise 1.) We thus have that all bridges of G are chords.

But we're nearly done now. If we know that a degree 3 graph G has a bridge graph that is an odd chordless j-cycle, and bridges which are chords, there is only one such graph G. (I'll let you work this out for yourself why this is true.) I've drawn the cases for $j = 3, 5, 7$ below. Now all we have to do is show that each of these graphs G contains a $K_{3,3}$. For the case where $j = 3$, this graph is itself a $K_{3,3}$. In the other two cases, I've highlighted the $K_{3,3}$ by labelling the points 1 or 2, depending on whether they're in the first or second set of vertices, and made the paths connecting them dashed, thicker, and a different color.

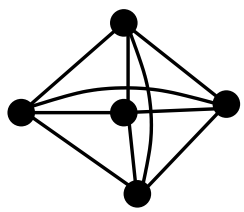


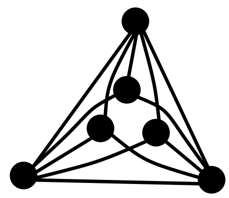
So now we've proved Kuratowski's theorem for graphs of maximum degree 3. What do we do for graphs of degree 4 or higher? If the chordless odd cycle has length five or more, the proof above that all the bridges are chords still works just fine. In this case, it is no longer true that we have just one graph corresponding to a j-cycle of the bridge; we now have a choice in constructing the graph. Namely, bridge i can either share a vertex with bridge $i + 2$ or not. We then discover that G looks something like one of the graphs below, and we have to show how to find a $K_{3,3}$ or a K_5 in any of these graphs. This is not too hard, and is an exercise below.

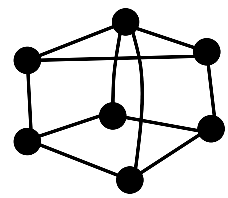


In the higher degree case, if the bridge graph is an odd cycle has length three, i.e. a triangle, things get tricky. We now have lots of graphs G where

you can't find a non-planar subgraph by replacing a bridge with a chord across two of its feet. Some of these are shown below. One thing that's fairly easy to show is that no bridge needs to have more than four feet (otherwise we could replace it with a smaller bridge connecting just some of its feet, and still keept it incompatible with the other two bridges). This observation reduces the proof to a finite number of cases, but the details are still quite tedious.







Is there a better way to organize these details? The best way I know of for dealing with this case is to use the fact that we can assume that C is the *longest* cycle in the graph. This means that for any bridge, there is a vertex on the cycle C between every pair of its feet. I'll let you work out the rest of this part of the proof (see exercises).

Exercise 1: Show that if G has degree 3, and the bridge graph is a triangle, then any non-chordal bridge can be replaced by a chord between two of its feet to get a new non-planar graph.

Doing the next five exercises will let you prove Kuratowski's theorem for the case where G has degree greater than 3. None of these exercises is very hard. Note that here, two bridges can have a foot at the same vertex.

Exercise 2: Show that any graph which has just chordal bridges, and where the bridge graph is an odd cycle of length ≥ 5, has a subgraph which is a subdivision of $K_{3,3}$ or K_5.

Exercise 3: Suppose we have a graph with three bridges B_1, B_2, and B_3, and the bridge graph is a triangle. Show that if B_1 has two feet which are adjacent on the cycle then there is a longer cycle in G.

Exercise 4: Suppose we have a graph with three bridges B_1, B_2, and B_3, and the bridge graph is a triangle. Show that if both B_2 and B_3 have a foot (these could be at the same vertex) between two adjacent feet of B_1 (adjacent around the cycle), then B_1 can be replaced by a chord between two of its feet to obtain a non-planar graph.

Exercise 5: Suppose we have a graph with at three bridges B_1, B_2, and B_3, and the bridge graph is a triangle. Show that if the cycle has four adjacent

vertices belonging to two bridges B_1, B_2, B_1, B_2, in that order, then graph G contains a longer cycle. (It doesn't matter if some of these vertices could also belong to B_3.)

Exercise 6: Suppose we have a graph with three bridges B_1, B_2, and B_3, the bridge graph is a triangle, and B_1 is not a chord. Show that if G is a minimal non-planar graph with no degree 2 vertices, and C is the longest cycle of G then exercises 3, 4, 5 imply that there is exactly one vertex on the cycle between any adjacent two feet of B_1, that B_2 and B_3 are chords, and that the feet must alternate as follows:

$$B_1, B_2, B_1, B_3, B_1, B_2, B_1, B_3.$$

Dealing with this last case is easy: do it.